

Parallel Flow Field Dynamics in the Sakoda Model of Social Interactions

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Abstract. We study a parallel flow field variant of the Sakoda model of social interactions, in which movement is local, synchronous, and constrained by a rotating directional field. We compare this dynamics with the sequential Sakoda rule across 35 interaction matrices with at least one attractive entry. In our experiments, the sequential dynamics reaches lower final energies in most paired simulations, while the parallel flow field dynamics produces higher energy states and a small number of atypical cases with a finite horizon. The results show how imposing local directional motion modifies the energy landscape explored by Sakoda-type social dynamics.

Keywords: Sakoda model · Cellular automata · Social dynamics

1 Introduction

The emergence of organized spatial structures from simple local interactions is a central question in social dynamics and complex systems [6]. Discrete dynamical models provide a useful framework for exploring how individual rules can generate collective spatial patterns.

In the late forties, Sakoda introduced a discrete dynamical model for social interaction [2, 3], around the same period as the segregation model proposed by Schelling [4, 5]. The Sakoda model generalizes this setting by allowing attraction, repulsion, and neutrality between two groups of individuals moving on a lattice with vacancies. Medina et al. [1] systematically characterized the resulting space of interactions, reducing the possible attitude matrices to 45 independent cases and classifying the corresponding social structures through an energy framework.

In the Sakoda dynamics studied by Medina et al. [1], individuals are selected sequentially: each selected individual evaluates all vacant sites in the lattice and moves to one that improves its satisfaction. Here, we take this dynamics as the reference case and compare it with a flow field dynamics, where movement is local and the update is synchronous. In the proposed scheme, each individual evaluates at most one local candidate destination indicated by a rotating directional field.

2 Model and dynamics

We consider a regular periodic lattice Λ with $N = L \times L$ sites. Each site $i \in \Lambda$ has a state $x_i \in \{0, 1, 2\}$, where 0 denotes a vacancy and 1, 2 denote the two types of individuals. The numbers of type 1 individuals, type 2 individuals, and vacancies are denoted by N_1 , N_2 , and N_V , respectively, and are preserved by the dynamics. We write $\phi_a = N_a/N$ for $a = 1, 2$ and $\phi_V = N_V/N$, with $\phi_1 + \phi_2 + \phi_V = 1$. Following the setting considered in [1], we restrict to $\phi_1 = \phi_2 = (1 - \phi_V)/2$.

The social interactions are summarized in a 2×2 matrix, denoted by S and represented as a vector $S = (s_{11}, s_{12}, s_{21}, s_{22})$, where $s_{\alpha\beta} \in \{+1, 0, -1\}$ is the attitude of individuals of type α toward those of type β , which can be attraction (+1), neutrality (0), or repulsion (-1). As in the Sakoda classification studied in [1], the $3^4 = 81$ possible matrices reduce to 45 non equivalent cases under exchange of type labels. Since the flow field dynamics produces no movement when all entries of S are non positive, we restrict the simulations to the 35 matrices containing at least one attractive interaction. We order these matrices using the scalar descriptor $\chi(S) = (s_{11} + s_{22}) - (s_{12} + s_{21})$ described in [1].

Let $x = (x_j)_{j \in \Lambda}$ be the current configuration of the lattice and $\mathcal{N}(i)$ the Von Neumann neighborhood of site i . For both dynamics, the local satisfaction of an individual occupying site i is defined as:

$$F_i(x_i; x) = \sum_{k \in \mathcal{N}(i)} \delta_S(x_i, x_k),$$

where

$$\delta_S(x_i, x_k) = \begin{cases} s_{\alpha\beta}, & \text{if } x_i = \alpha \neq 0 \wedge x_k = \beta \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For an occupied site i and a vacant site j in the current configuration x , let $x^{i \rightarrow j}$ denote the configuration obtained by moving the individual of type x_i from i to j . The gain associated with this candidate move is denoted as $\Delta F_{i \rightarrow j} = F_j(x_i; x^{i \rightarrow j}) - F_i(x_i; x)$. A move is allowed only if $\Delta F_{i \rightarrow j} > 0$.

For comparison between the two schemes, we distinguish between a step and a lattice update. In the sequential dynamics, a step refers to the evaluation of one selected occupied site, and a lattice update is completed once all occupied sites have been considered once in random order. In the parallel dynamics, there are no individual steps: one lattice update is a synchronous update in which all movement proposals are computed from the same configuration and the accepted moves are applied simultaneously.

2.1 Sequential Sakoda dynamics

We use as reference a sequential Sakoda dynamics with unrestricted destination search. At each step, an occupied site i is selected and the current configuration is used to evaluate $\Delta F_{i \rightarrow j}$ for every vacant site j in the lattice. If at least one vacancy gives a positive gain, the individual moves to a vacancy maximizing

$\Delta F_{i \rightarrow j}$; ties are resolved uniformly at random. The configuration is updated immediately after each move. The dynamics stops when a complete lattice update produces no movement, in which case the final configuration is declared stable, or when a prescribed maximum number of lattice updates is reached.

2.2 Parallel Sakoda dynamics with flow field

The second dynamics is based on the same local satisfaction rule, but restricts movement through a directional field. Each site i has an associated direction $d_i \in \{0, 1, 2, 3\}$, where the four values represent up, right, down, and left, respectively. During a lattice update, an occupied site i can only consider the neighboring site indicated by its current direction d_i ; denote this site by j . If j is not vacant, no proposal is made. If j is vacant and $\Delta F_{i \rightarrow j} > 0$, the individual proposes the move from i to j . All admissible movement proposals are computed from the same configuration. For each vacancy receiving several proposals, one of them is selected uniformly at random, while the remaining proposers stay at their current sites during that lattice update. All accepted moves are then applied simultaneously. If at least one proposal exists, each direction is rotated clockwise by 90° , $d_i \leftarrow (d_i + 1) \bmod 4$. The dynamics stops when no movement proposal exists under the current directional field, in which case the configuration is declared stable for the flow field dynamics, or when a prescribed maximum number of lattice updates is reached.

3 Experimental design

We compare the two dynamics introduced in Section 2 over the 35 interaction matrices retained in this study. For each matrix S , both dynamics are initialized from the same set of configurations, and the final state reached by each scheme is recorded, isolating the effect of replacing the sequential unrestricted movement rule with the parallel flow field dynamics.

All experiments are run on a square periodic lattice of side $L = 24$, with vacancy fraction $\phi_V = 1/2$ and balanced populations $\phi_1 = \phi_2 = 1/4$. For each interaction matrix S , we use five initial configurations $G_1, \dots, G_5$, generated by randomly placing the $N_1 + N_2$ individuals on one sublattice of a perfect chess pattern. This controlled setting keeps the lattice size fixed across all matrices, avoiding an additional finite size parameter in this exploratory study. Moreover, the chess initialization ensures that every occupied site has a vacant neighbor in each Von Neumann direction, so the flow field dynamics always admits at least one candidate destination in its first lattice update, and the initial energy is $E = 0$ for every S .

The initial flow field is generated once by assigning a direction sampled uniformly at random from $\{0, 1, 2, 3\}$ to each site, and is reused across all matrices and initial configurations. This treats the flow as a fixed external structure and isolates the effect of S . The dependence on the initial flow field is left for future work.

Figure 1 shows the five initial configurations used throughout the experiment, together with the initial directional field used by the flow field dynamics.

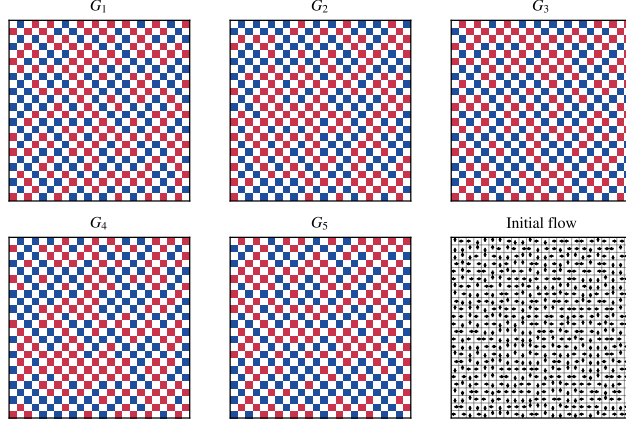


Fig. 1. The panels $G_1, \dots, G_5$ show the five initial lattice configurations: white sites are vacancies, blue sites are type 1 individuals, and red sites are type 2 individuals. All configurations follow the chess initialization, so occupied sites lie on one sublattice and initially have vacant Von Neumann neighbors. The last panel shows the fixed initial directional field used by the flow field dynamics; arrows indicate the direction assigned to each site. The same configurations and directional field are reused across all interaction matrices.

Each simulation is evolved until the corresponding dynamics reaches a stable state, as defined in Section 2, or until a cutoff of 500 lattice updates is reached. For each simulation, we retain the final configuration and its final energy.

The global energy of a configuration x is defined as:

$$E(x) = -\frac{1}{2} \sum_{i \in \Lambda: x_i \neq 0} F_i(x_i; x),$$

Since each site has four Von Neumann neighbors, the factor $\frac{1}{2} \cdot (1 - \phi_V)N \cdot 4 = 2(1 - \phi_V)N$ gives the natural energy scale associated with the maximal local interaction contribution. We therefore report the normalized energy:

$$E_{\text{norm}}(x) = \frac{E(x)}{2(1 - \phi_V)N}.$$

This normalization makes the energy comparable across configurations on a common scale. For nonsymmetric matrices S , the energy is not necessarily a Lyapunov function and may fluctuate during the evolution [1]. In the remainder of this study, all energy values refer to the normalized energy E_{norm} , unless explicitly stated otherwise.

4 Results

A first observation before comparing the two dynamics is that, for every interaction matrix S , the standard deviation of the final energy across $G_1, \dots, G_5$ remains below 0.033 for both dynamics. Thus, the final energy values are not strongly affected by the particular initial grid within the set considered here.

In Figure 2, we display the final configurations obtained from G_1 as representative snapshots of the flow field dynamics. Each lattice is placed at the value of $\chi(S)$ and at E_{norm} reached from that same initial grid. The figure provides a compact view of how the local flow rule organizes the interaction space. Matrices with strongly negative final energies appear both near the extreme values of χ and in highly attractive interaction structures, such as the fully attractive case $S = (1, 1, 1, 1)$, which reaches the lowest position in the diagram. A second group occupies intermediate energy levels, while the configurations with orange borders are those that did not reach a stable state before the cutoff. These cases lie in the upper part of the diagram and suggest longer transients or persistent activity under the flow field rule within the time horizon considered here.

We then compare these final states with those obtained from the sequential Sakoda dynamics. Across the 175 paired simulations, the sequential dynamics reaches a strictly lower final energy in 159 cases, the parallel flow field dynamics reaches a lower final energy in 10 cases, and the remaining 6 cases are ties. Thus, the dominant pattern is that the final energy of the parallel flow field dynamics exceeds that of the sequential Sakoda dynamics. To summarize how this gap depends on the interaction structure, Table 1 reports the mean normalized gap grouped by χ . The gap is not uniform across the interaction space: it is smaller near $\chi = 0$ and increases toward the extreme values of χ , especially on the positive side. The few reversals and near-ties occur in matrices for which the sequential dynamics reaches the cutoff in all five initial grids. For this reason, we interpret them as atypical cases with a finite horizon rather than as stable counterexamples to the dominant gap.³

To examine how the final energy differences arise during the evolution, we select four representative interaction matrices associated with the social structures described by Sakoda and later organized in the classification of Medina et al. [3, 1]. The selection includes the *boys and girls* case $S = (-1, 1, 1, -1)$, which is the opposite of segregation and corresponds to $\chi = -4$; the *inclusion* case $S = (1, 1, 1, 1)$, where all interactions are attractive; the *social workers* case, represented in our reduced list by the type exchanged case $S = (-1, -1, 1, 1)$; and the *segregation* case $S = (1, -1, -1, 1)$, where individuals are attracted to their own type and repelled by the other type. These matrices cover the two extreme values of χ and two qualitatively distinct cases at $\chi = 0$. For each matrix, Figure 3 shows the energy trajectory obtained from the representative initial

³ As a robustness check, the same analysis was repeated for $L = 12$. The qualitative picture was preserved: the direction of the energy gap remained the same in most paired simulations, and the atypical cases remained concentrated in the same region of the interaction space. We therefore report the $L = 24$ results in the main text.

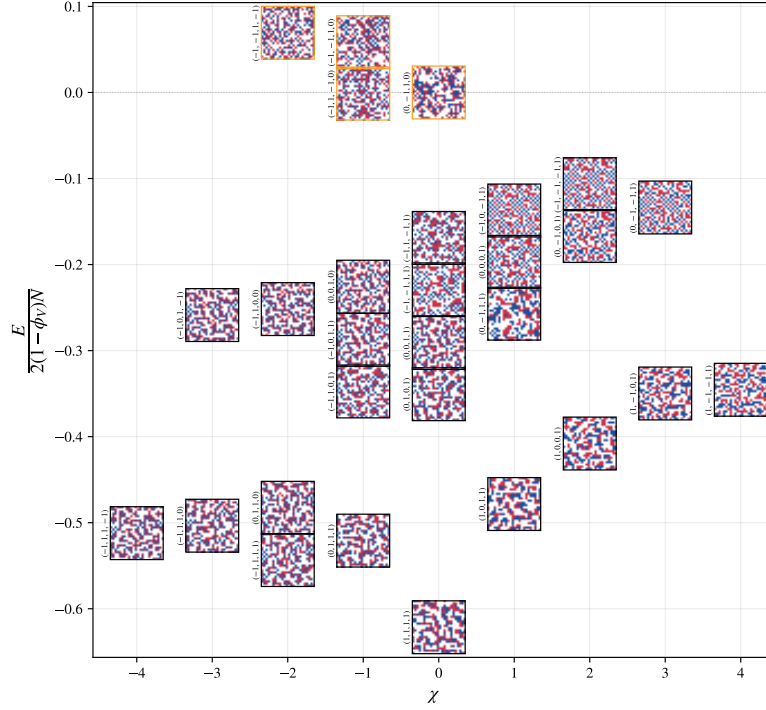


Fig. 2. Final configurations of the flow field dynamics in the χ - E_{norm} plane for $L = 24$ and $\phi_V = 1/2$, obtained from the initial grid G_1 . Borders distinguish the convergence behavior of the flow field dynamics: black for configurations that reached a stable state before the cutoff, orange for configurations that reached the cutoff of 500 lattice updates. For readability, six configurations with very close positions in the plane are not displayed: $S = (-1, 0, 1, 0)$, $S = (-1, 0, 0, 1)$, $S = (-1, -1, 0, 1)$, $S = (0, 1, -1, 1)$, $S = (0, 0, -1, 1)$, and $S = (1, -1, 1, 1)$.

grid G_1 under both the sequential Sakoda dynamics and the parallel flow field dynamics.

In three of the four cases, namely boys and girls, inclusion, and segregation, the sequential dynamics reaches a stable configuration in only a few lattice updates and attains a substantially lower final energy than the flow field dynamics. The flow field dynamics also decreases the energy, but it does so more gradually and stops at a higher energy level. This behavior is consistent with the previous comparison: the unrestricted movement rule allows the sequential dynamics to reach configurations with low energy more rapidly, whereas the local flow constraint slows down the energy decrease and limits the depth of the final state. The social workers case behaves differently. For $S = (-1, -1, 1, 1)$, the sequential dynamics does not stabilize within the cutoff and its energy fluctuates around an intermediate level, whereas the flow field dynamics reaches a stable state with

Table 1. Mean normalized energy gap grouped by χ . The gap is computed as the final normalized energy of the parallel flow field dynamics minus the final normalized energy of the sequential Sakoda dynamics, and then averaged over all paired simulations with the same value of χ .

χ	Matrices	Paired runs	Mean gap
-4	1	5	0.286
-3	2	10	0.198
-2	5	25	0.159
-1	6	30	0.104
0	7	35	0.083
1	6	30	0.149
2	5	25	0.286
3	2	10	0.319
4	1	5	0.403

lower final energy. This panel illustrates an atypical cases with a finite horizon identified above, where the sequential dynamics has not stabilized within the cutoff.

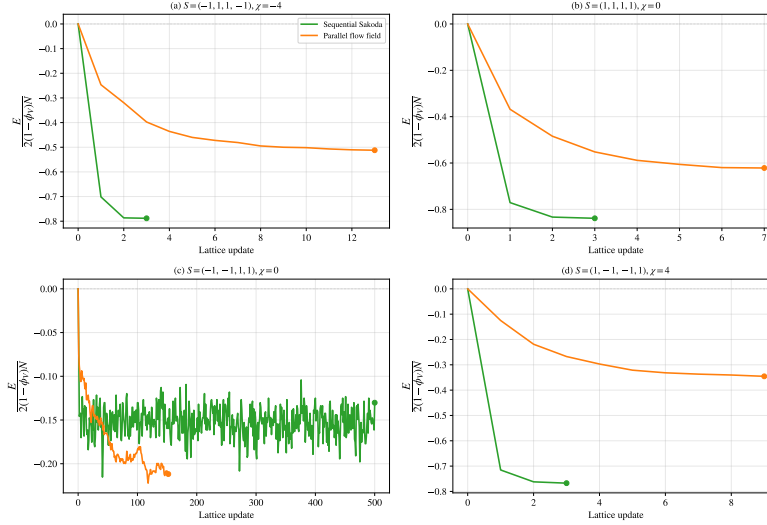


Fig. 3. Energy trajectories for four representative interaction matrices, using the initial grid G_1 on the $L = 24$ lattice with $\phi_V = 1/2$. Each panel compares the sequential Sakoda dynamics and the parallel flow field dynamics as functions of the number of lattice updates. All trajectories start from $E_{\text{norm}} = 0$ because the chess initialization gives vacant Von Neumann neighbors to every occupied site. The panels correspond to: (a) boys and girls, (b) inclusion, (c) social workers, and (d) segregation, $S = (1, -1, -1, 1)$, $\chi = 4$. Markers indicate the final recorded state of each trajectory, either a stable state or the state reached at the cutoff.

5 Conclusions

We introduced a parallel flow field dynamics as a local and synchronous variant of the sequential Sakoda dynamics. Both rules were compared over the 35 interaction matrices with at least one attractive entry, using $L = 24$, vacancy fraction $\phi_V = 1/2$, five chess initial configurations, a common initial flow field, and a cutoff of 500 lattice updates.

The dominant outcome is that the sequential Sakoda dynamics reaches lower final energy in most paired simulations, namely in 159 of the 175 runs. The few cases where the flow field dynamics reaches lower energy are concentrated among simulations in which the sequential dynamics does not stabilize within the cutoff.

This study remains exploratory, since several parameters are fixed. The comparison between $L = 12$ and $L = 24$ suggests qualitative stability across these two sizes, but not a systematic finite-size analysis. Likewise, using only $\phi_V = 1/2$, chess initial configurations, and one initial flow field provides a controlled setting for isolating the effect of the interaction matrix, while leaving robustness to lattice size, vacancy fraction, initial configurations, and imposed flow fields for future work. For non-stabilizing matrices, longer simulations are also needed to distinguish persistent activity from long transients.

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Code availability The source code used in this work is available in the GitHub repository `katerindelahoz/Sakoda_proyect`.

Disclosure of Interests. The authors have no competing interests to declare that are relevant to the content of this article.

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